Exhaustion of insert active material or heater failure problems have been nonexistent in the SERT II flight thrusters or ground life tests to date. This should continue to be so in future thrusters if correct operating temperatures and heater fabrication procedures are followed.

The reliability of the cathode heaters can be inferred from the consistency of the currents, voltages, and resistances (ratio of heating voltage-to-current). Reference 3 presents these values for all of the flight heaters, and the major variation ($\pm 3\%$) of the values were due to quantizing of the spacecraft data. Within this variation, the authors interpret the heater values to indicate no degradation of any SERT II heater.

### Summary

SERT II spacecraft data taken during the summer of 1973 indicated no starting degradation in any of the four hollow cathodes on board the spacecraft. Total hours of cathode operation for flight thrusters 1 and 2 were 3884 hr and 2165 hr with 144 and 188 restarts, respectively. Restarting was also accomplished after space storage periods up to 490 days. These data indicate that with proper design and operation a hollow cathode thruster will operate and restart for the long times and many restarts required by future missions.

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# Similarity Laws for Missiles of Minimum Ballistic Factor

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### Introduction

THE problem of determining the geometry of a slender axisymmetric body having a minimum ballistic factor has attracted considerable attention during the recent past.[1–5] Tawakley and Jain[6,7] have used the method of extremizing the product of powers of integrals to find the shapes of slender axisymmetric bodies having a minimum ballistic factor in Newtonian hypersonic flow when any two of the three quantities of length, diameter, and surface area are known in advance. In the case of minimizing the drag, Miele[8] has shown that a) a similarity law exists which enables determination of the optimum longitudinal contour of a body of arbitrary transversal contour from the known optimal longitudinal contour of a reference body, and b) a similarity law exists which enables determination of the optimum transversal contour of a body of arbitrary longitudinal contour from the known optimum transversal contour of a reference body. It is shown here that these similarity laws also exist for determining the minimum ballistic factor body shapes. The main assumptions are that the distribution of pressure coefficient is Newtonian, the skin-friction coefficient is constant, the body is slender in the longitudinal sense, and the body is homothetic (i.e., each cross section is geometrically similar to the base cross section and has the same orientation).

### Aerodynamic and Geometric Quantities

Let the shape of the body be represented in cylindrical coordinates $(x, r, \theta)$ by the equation

$$f(x, r, \theta) = 0 \tag{1}$$

where $x$ is the direction of the freestream, $r$ is the distance of any point from the $x$ axis and $\theta$ gives the angular position of this point with respect to some plane, and $\theta = 0$. Then, for a slender body in the longitudinal sense, the drag, the surface area, and the volume are given by[9]

$$\frac{D}{q} = \int_0^l \int_0^{2\pi} \frac{r}{f_r}\left[ -\frac{2f_x^{\,3}}{f_r^{\,2}+(f_\theta/r)^2} + C_f\,[f_r^{\,2}+(f_\theta/r)^2]^{1/2} dx\, d\theta \tag{2}$$

$$S = \int_0^l \int_0^{2\pi} \left(\frac{r}{f_r}\right)[f_r^{\,2}+(f_\theta/r)^2]^{1/2}\, dx\, d\theta \tag{3}$$

$$V = \frac{1}{2}\int_0^l \int_0^{2\pi} r^2 dx\, d\theta \tag{4}$$

where $q$ is the freestream dynamic pressure, $l$ is the length of the body, and $C_f$ is the constant skin-friction coefficient.

Since the body is supposed to be homothetic, Equation (1) is of the form

$$r = A(x)B(\theta) \tag{5}$$

where $A$ and $B$ denote arbitrary specified functions of $x$ and $\theta$, respectively. $A(x)$ describes the longitudinal contour and is such that $A(l) = 1$, whereas $B(\theta)$ describes the cross-sectional contour and is such that $B(0) = 1$. By making use of Eq. (5), the drag, the surface area, and the volume can be written as

$$\frac{D}{q} = 2\int_0^l A\dot{A}^3 dx \int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + C_f \int_0^l A dx \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{6}$$

$$S = \int_0^l A dx \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{7}$$

$$V = \frac{1}{2}\int_0^l A^2 dx \int_0^{2\pi} B^2 d\theta \tag{8}$$

where the dot represents the derivative with respect to the functional variable.

The ballistic coefficient of a body is proportional[2] to the ratio $D/qV$ which may be represented by $C$ (defined as "quality coefficient"). Therefore

$$C = \frac{D}{qV} = \frac{4\int_0^l A\dot{A}^3 dx \int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + 2C_f \int_0^l A dx \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta}{\int_0^l A^2 dx \int_0^{2\pi} B^2 d\theta} \tag{9}$$

The possible constraints can be on the drag, the length, the diameter or thickness, the surface area, and the volume, but

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the drag and the volume cannot be given simultaneously, since the ballistic factor would be known a priori and the variational problem would cease to exist.

**Similarity Law for Optimum Longitudinal Contour**

In this case the transversal contour is supposed to be prescribed, i.e., the function $B(\theta)$ is known. We define the following integrals:

$$J_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta; \quad J_2 = \frac{1}{2\pi}\int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta;$$

$$J_3 = \frac{1}{2\pi}\int_0^{2\pi} B^2 d\theta$$

so that the integrals $J_1$, $J_2$, and $J_3$ are known a priori. We also define the following modified drag, surface area, volume, skin-friction coefficient, and quality coefficient

$$\bar{D} = \frac{D}{J_1}, \quad \bar{S} = \frac{S}{J_2}, \quad \bar{V} = \frac{V}{J_3}, \quad \bar{C}_f = C_f \frac{J_2}{J_1}, \quad \bar{C} = C\frac{J_3}{J_1}$$

Therefore Eqs. (6–9) are replaced by

$$\frac{\bar{D}}{\pi q} = 4\int_0^l A\dot{A}^3 dx + 2\bar{C}_f \int_0^l A dx \tag{10}$$

$$\frac{\bar{S}}{\pi} = 2\int_0^l A dx \tag{11}$$

$$\frac{\bar{V}}{\pi} = \int_0^l A^2 dx \tag{12}$$

$$\bar{C} = \frac{4\int_0^l A\dot{A}^3 dx + 2\bar{C}_f \int_0^l A dx}{\int_0^l A^2 dx} \tag{13}$$

Now for a circular transversal contour $B(\theta) = 1$ everywhere. Therefore

$$J_1 = J_2 = J_3 = 1$$

which, based on Eqs. (6–9), implies that

$$\frac{D}{\pi q} = 4\int_0^l A\dot{A}^3 dx + 2C_f \int_0^l A dx \tag{14}$$

$$\frac{S}{\pi} = 2\int_0^l A dx \tag{15}$$

$$\frac{V}{\pi} = \int_0^l A^2 dx \tag{16}$$

$$C = \frac{4\int_0^l A\dot{A}^3 dx + 2C_f \int_0^l A dx}{\int_0^l A^2 dx} \tag{17}$$

Thus, the sets of Eqs. (10–13) and (14–17) are formally identical, implying the similarity law for the longitudinal contour that "the function $A(x)$ which optimizes the body of arbitrary but prescribed transversal contour is identical with the function $A(x)$ which optimizes the axisymmetric body provided the drag, the surface area, the volume, the coefficient of friction and the quality coefficient of the latter are replaced by appropriate proportional quantities of the former with the constants of proportionality depending upon the shape of the prescribed transversal contour." Because of this similarity law, the longitudinal contours of any arbitrary but prescribed transversal contour can be derived from the corresponding optimum shapes of axisymmetric bodies which have been derived in Refs. 6 and 7.

**Similarity Law for Transversal Contour**

In this case the longitudinal contour is supposed to be prescribed, i.e., the function $A(x)$ is known. We define the following integrals:

$$I_1 = 2l^2\int_0^l A\dot{A}^3 dx; \quad I_2 = \frac{2}{l}\int_0^l A dx; \quad I_3 = \frac{3}{l}\int_0^l A^2 dx$$

so that the integrals $I_1$, $I_2$, and $I_3$ are known a priori. We now introduce the following modified drag, surface area, volume, skin-friction coefficient and quality coefficient

$$\bar{D} = \frac{D}{I_1}, \quad \bar{S} = \frac{S}{I_2}, \quad \bar{V} = \frac{V}{I_3}, \quad \bar{C}_f = C_f \frac{I_2}{I_1}, \quad \bar{C} = C\frac{I_3}{I_1}$$

Therefore, Eqs. (6–9) are replaced by

$$\frac{\bar{D}l^2}{2} = \int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + \frac{l^3}{2}\bar{C}_f \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{18}$$

$$2\bar{S}l = \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{19}$$

$$\frac{6\bar{V}}{l} = \int_0^{2\pi} B^2 d\theta \tag{20}$$

$$\frac{\bar{C}l^3}{6} = \frac{\int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + \frac{l^3}{2}\bar{C}_f \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta}{\int_0^{2\pi} B^2 d\theta} \tag{21}$$

If the longitudinal contour is conical, i.e., $A(x) = x/l$ everywhere, then

$$I_1 = I_2 = I_3 = 1$$

which, based on Eqs. (6–9), implies that

$$\frac{Dl^2}{2} = \int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + \frac{l^3}{2}C_f \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{22}$$

$$2Sl = \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta \tag{23}$$

$$\frac{6V}{l} = \int_0^{2\pi} B^2 d\theta \tag{24}$$

$$\frac{Cl^3}{6} = \frac{\int_0^{2\pi} \frac{B^6}{B^2+\dot{B}^2} d\theta + \frac{l^3}{2}C_f \int_0^{2\pi} (B^2+\dot{B}^2)^{1/2} d\theta}{\int_0^{2\pi} B^2 d\theta} \tag{25}$$

Since the sets of Eqs. (18–21) and (22–25) are formally identical, implying the similarity law for the transversal contour that "the function $B(\theta)$ which optimizes the transversal contour of a body having an arbitrary but prescribed longitudinal contour is identical with the function $B(\theta)$ which optimizes transversal contour of a known longitudinal contour (say a conical body) provided the length of the two bodies is equal and the drag, the surface area, the volume, the coefficient of friction, and the quality coefficient of the latter are replaced by appropriate proportional quantities of the former with the constants of proportionality depending upon the shape of the known longitudinal contour."

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